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## INFORMATION ADVANTAGE IN COMMON-VALUE CLASSIC TULLOCK CONTESTS

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### Abstract

We show that in a common-value classic Tullock contests with incomplete information a player's information advantage is rewarded. Interestingly, in two-player contests both players exert the same expected effort. We characterize the equilibrium of two-player contests in which a player has information advantage, and show that this player exerts a larger effort and wins the price with a larger probability the larger is the realized value of the prize, although he wins the prize less frequently than his opponent. In addition, we find that players may exert more effort in a Tullock contest than in an all-pay auction.

**Keywords:** Information Advantage; Asymmetric Information; Common-Value; Tullock Contests.

**JEL Classification:** C72, D44, D82

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# 1 Introduction

Tullock contests (see Tullock 1980) are perhaps the most widely studied models in the literature on imperfectly discriminating contests. In a Tullock contest a player's probability of winning the prize is the ratio of the effort he exerts and the total effort exerted by all players. This paper belongs to a relatively recent but growing strand of this literature that concerns Tullock contests with incomplete information. Specifically, we study Tullock contests in which the players' common-value for the prize as well as their common marginal cost of effort depend on the state of nature. Players have a common prior belief, but upon the realization of the state of nature (and before taking action) each player observes some event that contains the realized state. The interim information endowment of each player at the moment of taking action is described by a partition of the set of states of nature, and these partitions may differ across players. (This representation is equivalent to the Harsanyi types model of Bayesian games, but is more natural in the present context.)

In this setting, we show that Tullock contests reward information advantages: if some player  $i$  has information advantage over another player  $j$  (i.e., the information partition of player  $i$  is finer than that of player  $j$ ), the expected payoff of player  $i$  is greater or equal to that of player  $j$ ; see Proposition 1. This result holds for any two players with rankable information partitions, regardless of the information endowments of the other players. The proof of this result relies on the theorem of Einy *et al.* (2002), which establishes that in any Bayesian Cournot equilibrium of an oligopolistic industry a firm's information advantage is rewarded provided the firms have linear costs. We establish the result by observing the formal equivalence between a Tullock contest and an oligopoly with asymmetric information.

We then proceed to study other properties of the equilibria of *two-player* contests. We first show that in such contests players exert the same expected effort in any equilibrium, provided their marginal costs is independent of the state; see Proposition 2. This result requires no relation between the players' information endowments. We then explicitly calcu-

late the unique Bayesian Nash equilibrium of a two-player contest in which one of the players has information advantage over his opponent; see Proposition 3. We find that the player with information advantage exerts a larger effort and wins the prize with a larger probability the larger is the realized value of the prize, although he wins the prize less frequently than his opponent; see Proposition 4. We also examine how the asymmetry of the information affects the players' efforts and their payoffs in relation to a symmetric scenario. Assuming that the distribution of the players' value for the prize is not disperse, we show that, when one player has information advantage over the other, the total effort exerted by the players is smaller, and thus the share of the total surplus they capture is larger, than when both players have the same information; see Proposition 5.

Our findings for two-player Tullock contests do not extend over to contests with more than two players. Indeed, we construct an example of three-player contest in which two of the players have the same information, which is superior to that of the third player, and the expected effort exerted by the last player is greater than those of his opponents. Also, we provide an example of a contest with multiple players in which a player that has information advantage over all the other players, who are symmetrically informed, wins the prize more frequently than any player. The linearity of the cost of effort is also crucial in our results regarding the reward of information advantages and the equality of expected efforts. Einy *et al.* (2002) provide an example showing that when firms cost function is state independent and quadratic (specifically,  $c(x) = x^2$ ) a firm with information advantage over other firm obtains lower expected profits. Interestingly, in any equilibrium of a two-player contest in which the state independent cost of effort is a function of the form  $c(x) = x^\alpha$  with  $\alpha \geq 1$ , the players' expected cost of effort coincide; see Proposition 6. Simple implications of this result are our Proposition 2, as well as the result that when  $\alpha > 1$  if a player with information advantage exerts less effort than his opponent; see Proposition 7.

Finally, we study the relative effectiveness of Tullock contests and all-pay auctions in

inducing the players to exert effort. Einy *et al.* (2015) characterize the unique equilibrium of a two-player common-value all-pay auction, which is in mixed strategies, and provide an explicit formula that allows us to compute the players' total effort. Using the formula in Einy *et al.* (2015) and results in the present work, we show that the sign of the difference in the total effort exerted by players in a Tullock contest and an all-pay auction can be either positive or negative.

There is an extensive literature on Tullock contests under complete information. Baye and Hoppe (2003) have identified a variety of economic settings (rent-seeking, innovation tournaments, patent races), which are strategically equivalent to a Tullock contest. Skaperdas (1996) and Clark and Riis (1998) provide axiomatic characterizations of Tullock contests. Perez-Castrillo and Verdier (1992), Baye Kovenock and de Vries (1994), Szidarovszky and Okuguchi (1997), Cornes and Hartley (2005), Yamazaki (2008) and Chowdhury and Sheremeta (2009) study existence and uniqueness of equilibrium. Skaperdas and Gan (1995), Glazer and Konrad (1999), Konrad (2002), Cohen and Sela (2005) and Franke *et al.* (2011) look into the effects of changes in the payoff structure on the behavior of players, and Schweinzer and Segev (2012) and Fu and Lu (2013) study optimal prize structures.

The study of Tullock contests under incomplete information is relatively sparse. Fey 2008 and Wasser 2011 study rent-seeking games under asymmetric information. Einy *et al.* (2015a) show that under standard assumptions Tullock contests with asymmetric information in a large class (which contains the contests studies in the present paper) have pure strategy Bayesian Nash equilibria, although they neither characterized equilibrium strategies nor they study their properties. Einy *et al.* (2016) study the value of public information.

A closely related paper by Warneryd (2003) studies two-player Tullock contests in which the players' common marginal cost of effort is state independent and their common value is a continuous random variable, and considers the alternative information structures arising when each player either observes the value or does not observe it. Warneryd (2003) shows

that in the equilibrium of the asymmetric information case, (1) the informed player is better off than the uninformed player, although (2) he wins the prize less frequently. Also, he shows that (3) both players' exert the same expected effort whether they are symmetrically or asymmetrically informed, and that (4) the expected total effort is strictly lower when players are asymmetrically informed than when they are symmetrically informed. The results we derive in our discrete setting contribute to clarify the nature of these results and explore their limits: Proposition 1 establishes that Tullock contests reward information advantages, which is the general idea implicit in (1), extending the result to contests with any number of players and arbitrary information structures whether the marginal cost of effort is state independent or not. Proposition 2 shows that in two-players Tullock contests the equality of players' expected efforts holds whatever the players information, implying (3) in our setting. Actually, this result is a simple corollary of a more general property of two-player Tullock contests: if players cost of effort is a state independent convex function of the form  $c(x) = x^\alpha$ , with  $\alpha \geq 1$ , then players expected cost of effort coincide; see Proposition 6. Thus, when the cost of effort is linear (i.e.,  $\alpha = 1$ ), this result implies (4), but when it strictly convex (i.e.,  $\alpha > 1$ ), a player with information advantage exerts less effort in expectation than his opponent; see Proposition 7. Also, propositions 4 and 5 establish in our setting the analog of (2) and (3), as well as other interesting properties of equilibrium.

As we do in the present paper, Fang (2002), Epstein, Mealem and Nitzan (2011) compare the outcomes of Tullock contests and all-pay auctions under complete information, while Dubey and Sahi (2012) consider an incomplete information binary setting. Common-value first- and second-price auctions in a setting analogous to ours have been studied by Einy *et al.* (2001, 2002), Forges and Orzach (2011), and Malueg and Orzach (2009, 2012), while all-pay auctions have been studied by Einy *et al.* (2015b, 2016) and Warneryd (2012).

The rest of the paper is organized as follows: Section 2 describes our setting and presents our result on information advantage. Section 3 contains our results for two-player contents.

Section 4 contains examples and discussion. The proofs are given in an Appendix.

## 2 Common-Value Classic Tullock Contests

### 2.1 Basic Notations

A group of players  $N = \{1, \dots, n\}$ , with  $n \geq 2$ , participates in a contest that allocates a prize. The contest starts by a move of nature that selects a state  $\omega$  from a *finite* set  $\Omega$ . The private information about the state of nature of player  $i \in N$  is described by a partition  $\Pi_i$  of  $\Omega$ . Upon the realization of  $\omega$ , player  $i$  observes the element  $\pi_i(\omega)$  of  $\Pi_i$  which contains  $\omega$ , i.e.,  $\pi_i(\omega)$  contains the set of states of nature which  $i$  cannot distinguish from  $\omega$ . Then players simultaneously choose their effort levels in  $\mathbb{R}_+$ . Players' common prior belief about the realized state of nature is given by a probability distribution  $p$  over  $\Omega$ . W.l.o.g. we assume that  $p(\{\omega\}) > 0$  for every  $\omega \in \Omega$ . Players' common value for the prize is given by a random variable  $V : \Omega \rightarrow \mathbb{R}_{++}$ . We assume that players' common costs of effort is linear; their common marginal cost of effort is given by a random variable  $C : \Omega \rightarrow \mathbb{R}_{++}$ . The prize is awarded to the players in a probabilistic fashion: Given a profile of efforts  $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n \setminus \{0\}$  the probability that player  $i$  receives the prize is equal to  $x_i / \sum_{k=1}^n x_k$ ; when no player exerts effort, i.e.,  $x = 0$ , the prize is allocated according to a given arbitrary probability vector  $(\bar{\rho}_1, \dots, \bar{\rho}_n)$ . Hence, for every  $\omega \in \Omega$  and  $x \in \mathbb{R}_+^n \setminus \{0\}$  the payoff of player  $i \in N$  is given by

$$u_i(\omega, x) = \frac{x_i}{\sum_{k=1}^n x_k} V(\omega) - C(\omega) x_i, \quad (1)$$

whereas  $u_i(\omega, 0) = \bar{\rho}_i V(\omega)$ . A *common-value classic Tullock contest* (which we will refer to henceforth simply as a *Tullock contest*) is therefore described by a collection

$$(N, (\Omega, p), \{\Pi_i\}_{i \in N}, V, C).$$

## 2.2 Strategies and Equilibrium

A Tullock contest defines a Bayesian game in which a pure strategy for player  $i \in N$  is a  $\Pi_i$ -measurable function  $X_i : \Omega \rightarrow \mathbb{R}_+$  (i.e.,  $X_i$  is constant on every element of  $\Pi_i$ ), that represents  $i$ 's choice of effort in each state of nature following the observation of his private information. We denote by  $S_i$  the set of strategies of player  $i$ , and by  $S = \times_{i=1}^n S_i$  the set of strategy profiles. For any strategy  $X_i \in S_i$  and  $\pi_i \in \Pi_i$ ,  $X_i(\pi_i)$  stands for the constant value that  $X_i(\cdot)$  takes on  $\pi_i$ . Also, given a strategy profile  $X = (X_1, \dots, X_n) \in S$ , we denote by  $X_{-i}$  the profile obtained from  $X$  by suppressing the strategy of player  $i \in N$ . Throughout the paper we restrict attention to pure strategies.

Let  $X = (X_1, \dots, X_n)$  be a strategy profile. We denote by  $U_i(X)$  the expected payoff of player  $i$ , which is given by

$$U_i(X) \equiv E[u_i(\cdot, (X_1(\cdot), \dots, X_n(\cdot)))].$$

For  $\pi_i \in \Pi_i$ , we denote by  $U_i(X \mid \pi_i)$  the expected payoff of player  $i$  conditional on  $\pi_i$ , i.e.,

$$U_i(X \mid \pi_i) \equiv E[u_i(\cdot, (X_1(\cdot), \dots, X_n(\cdot))) \mid \pi_i].$$

An  $n$ -tuple of strategies  $X^* = (X_1^*, \dots, X_n^*)$  is a (Bayesian Nash) equilibrium if

$$U_i(X^*) \geq U_i(X_{-i}^*, X_i) \tag{2}$$

for every player  $i \in N$ , and every strategy  $X_i \in S_i$ ; or equivalently,

$$U_i(X^* \mid \pi_i) \geq U_i(X_{-i}^*, x_i \mid \pi_i) \tag{3}$$

for every  $i \in N$ , every  $\pi_i \in \Pi_i$ , and every effort  $x_i \in \mathbb{R}_+$  of player  $i$  (viewed here as a strategy in  $S_i$  with the constant value  $x_i$  on the set  $\pi_i$ ). Existence of equilibrium in Tullock contests is implied by the theorem of Einy *et al.* (2015).

## 2.3 Information Advantage

The concept of information advantage is central to our work, and our first result is concerned with the natural question of whether this advantage is reflected in equilibrium payoffs. Formally, player  $i \in N$  is said to have an information advantage over player  $j \in N$  if partition  $\Pi_i$  is finer than partition  $\Pi_j$ , i.e.,  $\pi_i(\omega) \subset \pi_j(\omega)$  for every  $\omega \in \Omega$  – in words, if player  $i$  knows the realized state of nature with no less or greater precision than player  $j$ . The next proposition shows that Tullock contests reward information advantages, as far as the equilibrium expected payoffs are concerned – for any two players  $i$  and  $j$  as in the above scenario, the expected payoff of player  $i$  with an information advantage is never below that of the disadvantaged player  $j$ . Importantly, this result holds for any two players  $i$  and  $j$  with rankable information partitions, regardless of the information endowments of the other players and their relation to the information of  $i$  and  $j$ .

**Proposition 1** *In any equilibrium of a Tullock contest the expected payoff of a player is greater or equal to that of any other player over whom the player has information advantage.*

Proposition 1 is proved by observing a formal equivalence between a Tullock contest and a Cournot oligopoly with asymmetric information in which all firms have the same linear cost function, and by appealing to a result of Einy *et al.* (2002) that shows that the Bayesian Cournot equilibria of such industries have the desired property.

## 3 Two-Player Tullock Contests

### 3.1 Expected Effort in Equilibrium

Throughout Section 3 we will confine ourselves to *two-player* Tullock contests, and further assume that *the marginal cost of effort is state-independent* (i.e.,  $C$  is constant on  $\Omega$ ) and hence normalized to one. Our first result establishes the equality of the expected efforts of



both players in such contests. In particular, in a two-player Tullock contest in which one player has an information advantage over the other, the player with less information exerts the same expected effort as the player with information advantage, although his expected payoff is (typically) smaller than that of the better informed player by Proposition 1. The equality of the expected efforts does not, however, depend on there being an information advantage in the contest, as no assumption is made on the players' information endowments.

**Proposition 2** *In any equilibrium of a two-player Tullock contest both players exert the same expected effort.*

### 3.2 Equilibrium when a Player has Information Advantage

We shall now characterize the equilibrium of a two-player Tullock contest in which one player has an information advantage over the other, and study its properties. In order to simplify the presentation, let us index the set of states of nature as

$$\Omega = \{\omega_1, \dots, \omega_m\}.$$

For  $k = 1, \dots, m$ , we write

$$p(\{\omega_k\}) = p_k \text{ and } V(\omega_k) = v_k$$

and, w.l.o.g., assume that

$$0 < v_1 \leq v_2 \leq \dots \leq v_m. \tag{4}$$

We shall assume that player 2 has an information advantage over player 1. Since  $\Omega$  is finite, we may further assume, w.l.o.g., that the only information player 1 has about the state is the common prior belief, i.e.,  $\Pi_1 = \{\Omega\}$ , whereas player 2 has full information about the state of nature, i.e.,  $\Pi_2 = \{\{\omega_1\}, \dots, \{\omega_m\}\}$ . Dealing with general information structures for which player 2 has information advantage over player 1 involves applying on each atom of the information partition of player 1 the analysis that we provide for this extreme information

structure, and then use conditional expectation and the law of iterated expectation to derive the results we obtain.

With these conventions, a strategy profile is a pair  $(X, Y)$ , where  $X$  can be identified with a number  $x \in \mathbb{R}_+$  specifying player 1's unconditional effort, and  $Y$  can be identified with a vector  $(y_1, \dots, y_m) \in \mathbb{R}_+^m$  specifying the effort of player 2 in each of the  $m$  states of nature. The following notation will be useful in characterizing equilibria. For  $k \in \{1, \dots, m\}$  we write

$$A_k = \left( \sum_{s=k}^m p_s \sqrt{v_s} \right) \left( 1 + \sum_{s=k}^m p_s \right)^{-1}. \quad (5)$$

Note that

$$A_1 = \frac{E(\sqrt{V})}{2}.$$

Lemma 1 establishes a key property of the sequence  $\{A_k\}_{k=1}^m$ .

**Lemma 1** *If  $\sqrt{v_k} > A_k$ , then  $\sqrt{v_{k'}} > A_{k'}$  and  $A_k > A_{k'}$  for all  $k' > k$ .*

Proposition 3 below shows that a two-player Tullock contest in which player 2 has information advantage over player 1 has a unique equilibrium, which is calculated explicitly. Let  $k^* \in \{1, \dots, m\}$  be the smallest index such that  $\sqrt{v_k} > A_k$ . Since

$$\sqrt{v_m} > \frac{p_m}{(1 + p_m)} \sqrt{v_m} = A_m,$$

$k^*$  is well defined.

**Proposition 3** *A two-player Tullock contest in which player 2 has information advantage over player 1 has a unique equilibrium given by  $x^* = A_{k^*}^2$ ,  $y_k^* = 0$  if  $k < k^*$ , and  $y_k^* = A_{k^*} (\sqrt{v_k} - A_{k^*})$  otherwise.*

Proposition 3 implies uniqueness and symmetry of equilibrium in the complete information case, i.e., when  $m = 1$ . (Note that in this case  $k^* = 1$ , and therefore  $y_1^* = A_1(\sqrt{v_1} - A_1) = v_1/2 - v_1/4 = A_1^2 = x^*$ . This result is well known in the literature.)

We say that *the distribution of values is disperse* when  $\sqrt{v_1} \leq A_1 = E(\sqrt{V})/2$ ; e.g., this inequality holds when  $v_m \geq 4v_1$ . (We say that the distribution of values is *not* disperse if this inequality does not hold.) The following remark, which we will refer to later on, states that when the distribution of values is not disperse the unique equilibrium of a two-player Tullock contest in which one of the players has information advantage is interior, i.e.,  $k^* = 1$ , and makes precise a useful implication of propositions 1 and 3.

**Remark 1** *The unique equilibrium of a two-player Tullock contest in which player 2 has information advantage over player 1 is interior if and only if the distribution of values is not disperse. In an interior equilibrium the players expected efforts are  $E(X^*) = E(Y^*) = \left(E(\sqrt{V})\right)^2/4$ .*

In a two-player Tullock contest in which player 2 has information advantage over player 1 the equilibrium probability that player 1 wins the prize when the value is  $v_k$  is

$$\rho_{1k}^* = \frac{A_{k^*}^2}{A_{k^*}^2 + A_{k^*}(\sqrt{v_k} - A_{k^*})} = \frac{A_{k^*}}{\sqrt{v_k}}$$

when  $k \geq k^*$ , whereas the probability that player 2 wins the prize is  $\rho_{2k}^* = 1 - \rho_{1k}^*$ . Thus, the larger is the realized value of the prize, the smaller (larger) is the probability that player 1 (player 2) wins the prize, i.e.,  $\rho_{1k'}^* < \rho_{1k}^*$  and  $\rho_{2k'}^* > \rho_{2k}^*$  for  $k' > k \geq k^*$ . Of course, the larger is the realized value of the prize, the larger is the effort of player 2, i.e.,

$$y_{k'}^* = A_{k^*}(\sqrt{v_{k'}} - A_{k^*}) > A_{k^*}(\sqrt{v_k} - A_{k^*}) = y_k^*. \quad (6)$$

for  $k' > k \geq k^*$ . Additionally, for  $k' > k \geq k^*$ ,

$$\rho_{1k'}^* v_{k'} = A_{k^*} \sqrt{v_{k'}} > A_{k^*} \sqrt{v_k} = \rho_{1k}^* v_k,$$

i.e., the larger is the realized value of the prize, the larger is the conditional expected payoff of player 1; also,

$$\rho_{2k'}^* v_{k'} > \rho_{2k}^* v_{k'} > \rho_{2k}^* v_k,$$

i.e., the larger is the realized value of the prize, the larger is the conditional expected payoff of player 2. However, the ex-ante probability that player 1 wins the prize is larger than  $1/2$ , as we show in the proof of Proposition 4, which states these results.

**Proposition 4** *If  $m > 1$  and  $v_1 < v_m$ , then in the unique equilibrium of a two-player Tullock contest in which player 2 has information advantage over player 1, player 2 exerts a larger effort and wins the prize with a larger probability the larger is the realized value of the prize. However, the ex-ante probability that player 1 wins the prize is greater than that of player 2.*

The surplus captured by the players in a Tullock contest is the difference between the expected total surplus,  $E(V)$ , and the expected total effort they exert. By Remark 1, in a two-player Tullock contest in which values are not disperse and one of the players has information advantage the surplus captured by the players is  $E(V) - \left(E(\sqrt{V})\right)^2/2$ . Comparing the players expected efforts in this scenario and in the scenario in which players have symmetric information leads to an interesting observation, which we state in our last proposition of this section.

**Proposition 5** *Consider a two-player Tullock contest, and assume that  $m > 1$ ,  $v_1 < v_m$ , and that the distribution of values is not disperse. If one of the players has information advantage over the other, then the players exert less effort and capture a greater share of the surplus compared to the scenario where they are symmetrically informed.*

## 4 Discussion and Extensions

### 4.1 Illustration of the Results

The following example provides a simple demonstration of our general findings in Section 3.

**Example 1** *Consider a two-player Tullock contest with  $m = 2$ ,  $p_1 = 1 - p$ ,  $v_1 = 1$ , and  $v_2 = v$ , where  $p \in (0, 1)$  and  $v \in (1, \infty)$ . Then  $E(V) = 1 - p(1 - v)$ ,  $E(\sqrt{V}) = 1 - p(1 - \sqrt{v})$ ,*

$A_1 = E(\sqrt{V})/2$ , and  $A_2 = p\sqrt{v}/(1+p)$ . Assume that player 2 observes the value but player 1 does not observe it. If values are not disperse, i.e.,  $v < (1+p)^2/p^2$ , then the unique equilibrium is interior, and is given by

$$X^* = A_1^2, \quad Y^* = (A_1(1 - A_1), A_1(\sqrt{v} - A_1)).$$

The expected total effort is

$$TE^* = 2A_1^2 = [1 - p(1 - \sqrt{v})]^2/2.$$

We readily verify that

$$U_2(X^*, Y^*) - U_1(X^*, Y^*) = \frac{(1-p)p(1 - \sqrt{v})^2}{2} > 0,$$

that is, the payoff of player 2 is greater than that of player 1; and

$$E(\rho_1^*) = (1-p)A_1 + p\frac{A_1}{\sqrt{v}} = \frac{1}{2}(p + (1-p)\sqrt{v})\frac{1-p+p\sqrt{v}}{\sqrt{v}} \geq \frac{1}{1+p} > \frac{1}{2},$$

that is, the ex-ante probability that the player 2 wins the prize is less than that of player 1, consistently with propositions 1 and 4, respectively. Under symmetric information the equilibrium total effort is  $E(V)/2 > TE^*$ , i.e., when players have the symmetric information their expected total effort is larger than when player 2 has information advantage over player 1, consistently with Proposition 5.

If the values are disperse, i.e.,  $v \geq (1+p)^2/p^2$ , then the unique equilibrium is a corner equilibrium, given by

$$X^{**} = A_2^2, \quad Y^{**} = (0, A_2(\sqrt{v} - A_2)).$$

The expected total effort is

$$TE^{**} = 2A_2^2 = 2p^2v/(1+p)^2.$$

Again, we readily verify that

$$U_2(X^{**}, Y^{**}) - U_1(X^{**}, Y^{**}) = \frac{1-p}{2(p+1)}(p(v-1)-1) > \frac{1-p}{2p} > 0.$$

and

$$E(\rho_1^{**}) = (1 - p) + p \frac{A_2}{\sqrt{v}} = (1 - p) + \frac{p^2}{1 + p} = \frac{1}{1 + p} > \frac{1}{2}.$$

## 4.2 Contests with more than Two Players

Our next example shows that in a three player Tullock contest the players equilibrium expected efforts may differ, and hence that Proposition 2 does not extend to Tullock contests with more than two players.

**Example 2** Consider a three-player Tullock contest in which  $m = 2$ ,  $p_1 = p_2 = 1/2$ ,  $v_1 = 1$  and  $v_2 = 2$ . Assume that players 2 and 3 have information advantage over player 1, i.e.,  $\Pi_1 = \{\{\omega_1, \omega_2\}\}$ , and  $\Pi_2 = \Pi_3 = \{\{\omega_1\}, \{\omega_2\}\}$ . The equilibrium of this contest, which is interior, is readily calculated. In equilibrium, the strategy of strategy of player 1 is  $X_1^* = (0.30899, 0.30899)$  while the strategies of players 2 and 3 are  $X_2^* = X_3^* = (0.20342, 0.46933)$ . Hence

$$E(X_1^*) = 0.30899 < \frac{1}{2}(0.20342 + 0.46933) = E(X_2^*) = E(X_3^*),$$

i.e., the expected effort of player 1 is less than those of players 2 and 3.

Next we present an example of Tullock contests in which a player that has information advantage over the other players wins the prize more frequently than these players, which shows that Proposition 4 does not extend to Tullock contests with more than two players.

**Example 3** Consider the Tullock contest described in Example 2, but assume that there are 8 players, and that only player 8 observes the value, i.e.,  $\Pi_i = \{\{\omega_1, \omega_2\}\}$  for  $i \in \{1, \dots, 7\}$ , and  $\Pi_8 = \{\{\omega_1\}, \{\omega_2\}\}$ . The unique equilibrium of this contest is given by

$$X_1^* = \dots = X_7^* = (0.15551, 0.15551), \quad X_8^* = (0, 0.38694),$$

Thus, the ex-ante probability that player  $i \in \{1, 2, \dots, 7\}$  wins the prize is

$$E(\rho_i^*) = \frac{1}{2} \left( \frac{1}{7} + \frac{0.15551}{7(0.15551) + 0.38694} \right) = 0.12413,$$

whereas the ex-ante probability that player 8 win the prize is

$$E(\rho_8^*) = 1 - 7(0.12413) = 0.13109,$$

i.e., the player with information advantage wins the prize more frequently than his opponents.

### 4.3 Non-linear Costs

The linearity of the cost function is a crucial assumption in our results. Einy *et al.* (2002) provide an example showing of a duopoly in which firms state independent cost function is  $c(x) = x^2$  in which the expected payoff of a firm with information advantage is less than that of its competitor. In fact, in our setting we show that when players' cost of effort is a state-independent state independent cost function of the class  $c(x) = x^\alpha$  with  $\alpha \geq 1$  then in any equilibrium of a two-player Tullock contest players' cost of effort coincide. This result, which state below, implies Proposition 2 dealing with the case  $\alpha = 1$ , and also has implications over the relation of players efforts when a player has information advantage and  $\alpha > 1$ .

**Proposition 6** *In any equilibrium of a two-player Tullock contest in which the players' cost of effort is state-independent and given by the function  $c(x) \equiv x^\alpha$  with  $\alpha \geq 1$  the players' expected costs of effort coincide.*

Consider a two-player Tullock contest in which the players' cost of effort is state-independent and given by the function  $c(x) \equiv x^\alpha$  with  $\alpha > 1$ . An equilibrium of this contest  $(X, Y)$ , which exist by the theorem of Einy *et al.* (2015a), is identified by the effort levels  $x$  and  $(y_1, \dots, y_m)$ , respectively. By Proposition 6 the players' expected costs of effort coincide, i.e.,  $E(X^\alpha) = E(Y^\alpha)$ . Since  $x^\alpha = E(X^\alpha)$ , then  $v_1 > v_m$  implies that  $y_1 \neq y_m$  and therefore

$$x = E(X^\alpha)^{\frac{1}{\alpha}} = E(Y^\alpha)^{\frac{1}{\alpha}} > E(Y),$$

i.e., the player with information advantaged exerts *more* effort than his opponent. We state this result below.

**Proposition 7** *Consider a two-player Tullock contest in which the players' cost of effort is state-independent and given by a function  $c(x) \equiv x^\alpha$  with  $\alpha > 1$ . If  $m > 1$ ,  $v_1 < v_m$ , and player 2 has information advantage over player 1, then in any equilibrium of the contest the expected effort of player 1 is greater than that of player 2.*

#### 4.4 Comparison with All-Pay Auctions

Finally, we discuss whether the players' expected total effort in a Tullock contests can be ranked relative to that in an all-pay auction. Many applications of contest theory to political or sport competition, patent races, etc., model agents interactions as all-pay auctions. In a common-value all-pay auction the prize is given to player who exerts the largest effort. Einy *et al.* (2015b) consider a setting identical to that described in section 4, and show that in the unique equilibrium of a two-player (common-value) all-pay auction in which player 2 has information advantage over player 1 the players' total expected effort is

$$TE^{APA} = 2 \sum_{s=1}^m p_s \sum_{k=1}^{s-1} p_k v_k + \sum_{s=1}^m p_s^2 v_s.$$

Let us use this formula to compare total expected effort in two-player all-pay auction and Tullock contests in which player 2 has information advantage over player 1 in the environment described in Example 1. If values are not disperse,  $v < (1+p)^2/p^2$ , then

$$\begin{aligned} TE^{APA} - TE &= 2(1-p)p + (1-p)^2 + p^2v - 2\frac{(p\sqrt{v})^2}{(1+p)^2} \\ &= 2(1-p)p + \frac{1}{2}(1-p-p\sqrt{v})^2 \\ &> 0. \end{aligned}$$

Hence, the all-pay auction generates more effort than the Tullock contest. (We can show that this is generally the case when  $m = 2$  and values are not disperse.) However, if values are disperse, i.e.,  $v \geq (1+p)^2/p^2$ , the unique equilibrium of the Tullock contest is a corner



equilibrium, and

$$\begin{aligned} TE^{APA} - TE &= 2(1-p)p + (1-p)^2 + p^2v - 2\frac{(p\sqrt{v})^2}{(1+p)^2} \\ &= (1-p)(1+p) - p^2v \left( \frac{2}{(1+p)^2} - 1 \right). \end{aligned}$$

This difference is negative for, e.g.,  $p = 1/4$  and  $v > 375/7$ . Therefore, in general the level of effort generated by these two contests cannot be ranked.

## 5 Appendix

**Proof of Proposition 1.** An  $n$ -player Tullock contest  $(N, (\Omega, p), (\Pi_i)_{i \in N}, V, C)$  is formally identical to an oligopolist industry  $(N, (\Omega, p), (\Pi_i)_{i \in N}, P, C)$ , where the market demand  $P$  is defined for  $(\omega, x) \in \Omega \times \mathbb{R}_{++}$  as

$$P(\omega, x) = \frac{V(\omega)}{x}.$$

With this notation, the state-dependent profit of firm  $i \in N$  in the industry coincides with the payoff of player  $i \in N$  in the contest, i.e., for  $\omega \in \Omega$  and a non-zero  $X \in S$ ,

$$\begin{aligned} u_i(\omega, X) &= \frac{V(\omega)}{\sum_{s=1}^n X_s(\omega)} X_i(\omega) - C(\omega) X_i(\omega) \\ &= P(\omega, \sum_{s=1}^n X_s(\omega)) X_i(\omega) - C(\omega) X_i(\omega). \end{aligned}$$

Proposition 1 then follows from the theorem of Einy *et al.* (2002). ■ (It is easy to show that in any equilibrium  $X$  of a Tullock contest the total effort is positive in all states of nature. Thus, although the demand function  $P(\omega, 0)$  is not defined, and therefore does not formally satisfy the assumptions of Einy *et al.* (2002), their proof applies to this case without change.)

**Proof of Proposition 2.** Follows from Proposition 6 – see below.

**Proof of Lemma 1.** Assume that  $\sqrt{v_{\bar{k}}} > A_{\bar{k}}$  for some  $\bar{k} < m$ .

We show that  $\sqrt{v_k} > A_k$  for all  $k > \bar{k}$ . Suppose not; let  $\hat{k} > \bar{k}$  be the first index  $k > \bar{k}$  such that for  $\sqrt{v_k} \leq A_k$ . Note that  $v_{\hat{k}} \geq v_{\hat{k}-1}$  and  $\sqrt{v_{\hat{k}-1}} > A_{\hat{k}-1}$  imply

$$\begin{aligned} \left(1 + \sum_{s=\hat{k}}^m p_s\right) \sqrt{v_{\hat{k}}} &\geq \left(1 + \sum_{s=\hat{k}-1}^m p_s\right) \sqrt{v_{\hat{k}-1}} - p_{\hat{k}-1} \sqrt{v_{\hat{k}-1}} \\ &> \left(1 + \sum_{s=\hat{k}-1}^m p_s\right) A_{\hat{k}-1} - p_{\hat{k}-1} \sqrt{v_{\hat{k}-1}} \\ &= \sum_{s=\hat{k}-1}^m p_s \sqrt{v_s} - p_{\hat{k}-1} \sqrt{v_{\hat{k}-1}} \\ &= \left(1 + \sum_{s=\hat{k}}^m p_s\right) A_{\hat{k}}, \end{aligned}$$

which contradicts the assumption that  $\sqrt{v_{\hat{k}}} \leq A_{\hat{k}}$ .

Now we show that  $A_{\tilde{k}} > A_k$  for all  $k > \bar{k}$ . Suppose not; let  $\tilde{k} > \bar{k}$  be the first index  $k > \bar{k}$  such that  $A_{\tilde{k}} \leq A_k$ . Since  $\sqrt{v_{\tilde{k}-1}} > A_{\tilde{k}-1}$  (as we have just shown),

$$\begin{aligned} \left(1 + \sum_{s=\tilde{k}-1}^m p_s\right) A_{\tilde{k}-1} &= \sum_{s=\tilde{k}-1}^m p_s \sqrt{v_s} \\ &= p_{\tilde{k}-1} \sqrt{v_{\tilde{k}-1}} + \sum_{s=\tilde{k}}^m p_s \sqrt{v_s} \\ &> p_{\tilde{k}-1} A_{\tilde{k}-1} + \left(1 + \sum_{s=\tilde{k}}^m p_s\right) A_{\tilde{k}}. \end{aligned}$$

Hence

$$\left(1 + \sum_{s=\tilde{k}-1}^m p_s\right) A_{\tilde{k}-1} - p_{\tilde{k}-1} A_{\tilde{k}-1} > \left(1 + \sum_{s=\tilde{k}}^m p_s\right) A_{\tilde{k}},$$

i.e.,

$$\left(1 + \sum_{s=\tilde{k}}^m p_s\right) A_{\tilde{k}-1} > \left(1 + \sum_{s=\tilde{k}}^m p_s\right) A_{\tilde{k}}.$$

Thus,  $A_{\tilde{k}} \geq A_{\tilde{k}-1} > A_{\tilde{k}}$ , which contradicts the choice of  $\tilde{k}$ . ■

**Proof of Proposition 3.** Let  $(X, Y)$ , where  $X = x$  and  $Y = (y_1, \dots, y_m)$ , be an equilibrium. (Existence of equilibrium is guaranteed by the theorem of Einy et al (2015a).) If  $x = 0$  and

$Y = 0$ , then the prize is allocated with probabilities  $(\bar{\rho}_1, \bar{\rho}_2)$ , and the player who gets the prize with probability less than 1 can profitably deviate by exerting an arbitrarily small effort  $\varepsilon > 0$ . If  $x = 0$  and  $Y \neq 0$ , i.e.,  $y_k > 0$  for some  $k \in \{1, \dots, m\}$ , then the prize is allocated to player 2 at  $\omega_k$ , who can profitably deviate by reducing his effort to  $y_k/2$ . Likewise, if  $x > 0$  and  $Y = 0$ , then the prize is allocated to player 1, who can profitably deviate by reducing his effort to  $x/2$ . Hence  $x > 0$  and  $y_k > 0$  for some  $k \in \{1, \dots, m\}$ .

Since  $x > 0$  maximizes player 1's payoff given  $Y$ ,

$$\frac{\partial}{\partial x} \left( \sum_{s=1}^m p_s \left( v_s \frac{x}{x + y_s} - x \right) \right) = \sum_{s=1}^m p_s v_s \frac{y_s}{(x + y_s)^2} - 1 = 0. \quad (7)$$

And since  $y_s$  maximizes player 2's payoff in state  $\omega_s$  given  $x$ ,

$$\frac{\partial}{\partial y_s} \left( v_s \frac{y_s}{x + y_s} - y_s \right) = v_s \frac{x}{(x + y_s)^2} - 1 \leq 0, \quad (8)$$

(with equality if  $y_s > 0$ ) for each  $s = 1, \dots, m$ .

Notice next that if  $y_k > 0$  for some  $k < m$ , then  $y_{k'} > 0$  for all  $k' > k$ . Since  $x > 0$ , if  $y_k > 0$  then  $y_k = \sqrt{x} (\sqrt{v_k} - \sqrt{x})$  by (8), and since  $v_{k'} \geq v_k$  for all  $k' > k$ ,  $\sqrt{x} (\sqrt{v_{k'}} - \sqrt{x}) > 0$ , i.e.,

$$v_{k'} \frac{x}{x^2} - 1 > 0,$$

for all  $k' > k$ . Then  $y_{k'} = 0$  would violate inequality (8) for  $s = k'$ . Hence  $y_{k'} > 0$ .

Let  $k^\circ$  be the smallest index such that  $y_k > 0$ . Thus, (7) implies

$$\sum_{s=1}^m p_s v_s \frac{y_s}{(x + y_s)^2} = \sum_{s=k^\circ}^m p_s v_s \frac{y_s}{(x + y_s)^2} = 1,$$

and (8) implies  $y_{k'} = \sqrt{x} (\sqrt{v_{k'}} - \sqrt{x}) > 0$  for all  $k' \geq k^\circ$ . Hence  $x = A_{k^\circ}^2$ ,  $y_k = A_{k^\circ} (\sqrt{v_k} - A_{k^\circ})$  for all  $k \geq k^\circ$ , and  $y_k = 0$  for all  $k < k^\circ$ .

We now show that  $k^\circ = k^*$ , which establishes that the profile  $(x^*, y_1^*, \dots, y_m^*)$  identified in Proposition 3 is the unique equilibrium. Assume first that  $k^\circ < k^*$ . Then  $\sqrt{v_{k^\circ}} \leq A_{k^\circ}$  since  $k^*$  is the smallest index such that  $\sqrt{v_k} > A_k$ , and hence  $y_{k^\circ} = \sqrt{x} (\sqrt{v_{k^\circ}} - \sqrt{x}) =$

$A_{k^\circ} (\sqrt{v_{k^\circ}} - A_{k^\circ}) \leq 0$ , a contradiction as  $y_{k^\circ} > 0$  by the definition of  $k^\circ$ . Assume next that  $k^\circ > k^*$ . In this case,  $y_{k^*} = 0$ . Since  $\sqrt{v_{k^*}} > A_{k^*}$ , by Lemma 1

$$A_{k^*}^2 > A_{k^\circ}^2 = x, \quad (9)$$

and therefore

$$v_{k^*} \frac{x}{x^2} - 1 = \frac{A_{k^\circ}^2}{A_{k^\circ}^4} (v_{k^*} - A_{k^\circ}^2) > 0.$$

This stands in contradiction to (8), as  $y_{k^*} = 0$  by the definition of  $k^\circ (> k^*)$ . We conclude that indeed  $k^\circ = k^*$ . ■

**Proof of Proposition 4.** Let us be given a two-player Tullock contest in which player 2 has information advantage over player 1. It only remains to show that the ex-ante probability that player 1 wins the prize,  $E(\rho_1)$ , is greater than that of player 2,  $E(\rho_2)$ . By Proposition 3, the unique equilibrium of the contest is identified by some index  $k^* \in \{1, \dots, m\}$  and a vector  $(x^*, y_{k^*}^*, \dots, y_m^*) \gg 0$ . For  $(y_{k^*}, \dots, y_m) \in \mathbb{R}_+^{m-k^*+1}$  define the function

$$\bar{p}_2(y_{k^*}, \dots, y_m) := \sum_{k=k^*}^m \frac{p_k y_k}{y_k + \sum_{s=k^*}^m p_s y_s}.$$

By propositions 2 and 3

$$x^* = E(X^*) = E(Y^*) = \sum_{s=k^*}^m p_s y_s^*.$$

Hence

$$E(\rho_2) = \sum_{k=k^*}^m p_s \frac{y_k^*}{y_k^* + x^*} = \bar{p}_2(y_{k^*}^*, \dots, y_m^*).$$

We show that a maximum point  $\bar{y}$  of  $\bar{p}_2$  on  $K = \{(y_{k^*}, \dots, y_m) \in \mathbb{R}_+^{m-k^*+1} \mid y_{k^*} \leq y_{k^*+1} \leq \dots \leq y_m\}$  must satisfy  $\bar{y}_{k^*} = \dots = \bar{y}_m$ , and hence that

$$\max_K \bar{p}_2 = \frac{\sum_{s=k^*}^m p_s}{1 + \sum_{s=k^*}^m p_s} \leq \frac{1}{2}. \quad (10)$$

As shown above, see equation (6),  $y_{k^*}^* < \dots < y_m^*$ ; then (10) implies

$$E(\rho_2) = \bar{p}_2(y_{k^*}^*, \dots, y_m^*) < \max_K \bar{p}_2 \leq 1/2,$$

i.e., player 2 wins the prize with a smaller ex ante probability than player 1.

Differentiating  $\bar{p}_2$  with respect to  $y_k$  for  $k \in \{k^*, \dots, m\}$  we get

$$\frac{\partial \bar{p}_2}{\partial y_k} = p_k \left( \sum_{t=k^*, t \neq k}^m \frac{p_t y_t}{(y_k + \sum_{s=k^*}^m p_s y_s)^2} - \sum_{t=k^*, t \neq k}^m \frac{p_t y_t}{(y_t + \sum_{s=k^*}^m p_s y_s)^2} \right). \quad (11)$$

For every  $(y_{k^*}, \dots, y_m) \in K$  such that  $y_{k^*} < y_{k^*+1} \leq \dots \leq y_m$ ,  $\partial \bar{p}_2 / \partial y_{k^*}(y) > 0$ , and therefore necessarily  $\bar{y}_{k^*} = \bar{y}_{k^*+1}$ . Suppose now that it has already been shown that  $\bar{y}_{k^*} = \bar{y}_{k^*+1} = \dots = \bar{y}_k$ ,  $m-1 \geq k > 1$ . We show that  $\bar{y}_{k+1} = \bar{y}_k$  as well. Indeed, if  $\bar{y}_{k^*} = \bar{y}_{k^*+1} = \dots = \bar{y}_k < \bar{y}_{k+1} \leq \dots \leq \bar{y}_m$ , then by (11) we obtain that  $\partial \bar{p}_2 / \partial y_k(\bar{y}) > 0$ , a contradiction. Thus  $\bar{y}_{k^*} = \dots = \bar{y}_m$ . ■

**Proof of Proposition 5.** By Remark 1, in a two-player Tullock contest in which values are not disperse and player 2 has information advantage over player 1 the expected total effort is  $(E(\sqrt{V}))^2/2$ . When players have symmetric information the expected total effort is  $E(V)/2$ . If  $m > 1$ , then  $v_1 < v_m$  together with Jensen's inequality imply

$$\frac{E(V)}{2} > \frac{(E(\sqrt{V}))^2}{2}. \quad \blacksquare$$

**Proof of Proposition 6.** Note that, for every  $\omega \in \Omega$ ,  $\max(X_1^*(\omega), X_2^*(\omega)) > 0$ , since otherwise one of the players (say,  $i$ ) would have a profitable deviation at  $\pi_i(\omega)$  to an effort that is slightly above zero. Thus  $X_i^*(\pi_i)$  solves

$$\max_{x_i \in \mathbb{R}_+} E \left( \frac{V}{x_i + X_j^*} x_i - x_i^\alpha \mid \pi_i \right),$$

where  $j$  denotes  $i$ 's opponent. It follows that

$$\left. \frac{\partial}{\partial x_i} E \left( \frac{V}{x_i + X_j^*} x_i - x_i^\alpha \mid \pi_i \right) \right|_{x_i = X_i^*(\pi_i)} = E \left( \frac{V \cdot X_j^*}{(X_i^*(\pi_i) + X_j^*)^2} - \alpha x_i^{\alpha-1} \mid \pi_i \right) \leq 0,$$

where the inequality holds with equality if  $X_i^*(\pi_i) > 0$ . Hence

$$E \left( \frac{V \cdot X_i^*(\pi_i) \cdot X_j^*}{\alpha (X_i^*(\pi_i) + X_j^*)^2} - X_i^*(\pi_i)^\alpha \mid \pi_i \right) = \frac{X_i^*(\pi_i)}{\alpha} \cdot E \left( \frac{V \cdot X_j^*}{(X_i^*(\pi_i) + X_j^*)^2} - \alpha X_i^*(\pi_i)^{\alpha-1} \mid \pi_i \right) = 0,$$

and therefore

$$\sum_{\pi_i \in \Pi_i} E \left( \frac{V \cdot X_i^*(\pi_i) \cdot X_j^*}{\alpha (X_i^*(\pi_i) + X_j^*)^2} - X_i^*(\pi_i)^\alpha \mid \pi_i \right) \cdot p(\pi_i) = E \left( \frac{V \cdot X_i^* \cdot X_j^*}{\alpha (X_i^* + X_j^*)^2} \right) - E(C(X_i^*)) = 0.$$

It follows that

$$E(C(X_1^*)) = E \left( \frac{V \cdot X_1^* \cdot X_2^*}{\alpha (X_1^* + X_2^*)^2} \right) = E \left( \frac{V \cdot X_2^* \cdot X_1^*}{\alpha (X_2^* + X_1^*)^2} \right) = E(C(X_2^*)). \blacksquare$$

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